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Simple Model for Turbulent Tip Vortices

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Introduction

VORTICES are interwoven into the fabric of fluid mechanics transporting mass momentum and energy in the majority of natural and industrial flows. In technology these are either produced deliberately to accomplish the task, improve the function of devices, or emerge as a parasitic by-product of fluid motion. Aeronautical applications are very susceptible to these flow manifestations. Consequences of lift, tip vortices cause considerable drag, noise, and/or hazard in fixed- and/or rotating-wing aircraft. Although complicated in nature simple mathematical models are routinely used to elaborate on some of their fundamental properties [1,2].

It is a well-known fact that, irrespective of the host flowfield, intense vortices are analogous. Theoretically, intense (or strong) vortices are those where the tangential velocity component is orders of magnitude larger than the radial and axial. In practice, however, the property is also maintained by vortices where the swirl velocity component is dominant, but not necessarily of a magnitude enormously greater than the other two. Under these conditions the vortex appears to develop in a manner that the swirling action ignores the secondarylike flow in the azimuthal plane. Consequently, simple Rankine-like formulations have been used widely to model a variety of geophysical, wingtip, cyclone chamber, ship propeller, intake, and other types of vortices.

The physics of laminar vortices is relatively well known. Simple exact solutions of the Navier–Stokes equations are due to Rankine [3], Oseen [4], Lamb [5], Burgers [6], and others. Every one of them represents a possible solution, applicable to low vortex Reynolds numbers (Re_ν) defined as the total circulation divided by the kinematic viscosity. In antithesis, our grasp reduces exponentially for turbulent vortices. Certainly this is not accidental. Turbulent flows are considerably more complex both analytically and experimentally. Even today the turbulent vortex is among the not very well-explored territories in aerodynamics.

Not long ago, Ramasamy and Leishman [7] examined turbulent helicopter tip vortices using state of the art instruments and experimental techniques. Their high-resolution visualizations revealed that these types of whirls display the already familiar path to turbulence (probably slow, through spectral development). Inside the core, they have confirmed that helicopter tip vortices do enjoy laminarlike conditions. Approaching the core from the origin, past the laminar core and when a critical local Reynolds number is reached, the flow enters a transition region. This condition persists until a second critical Reynolds number where the flowfield changes into the "turbulent state" at larger radii. Most important of all, the accompanied high-fidelity velocity data (at $Re_\nu = 48,000$) exposed the following particular behavior for the azimuthal velocity component. In the region where the vortex is turbulent, the velocity decreases at a rate noticeably smaller than that of a laminar vortex. As a consequence of the new experimental evidence it became amply evident that high Re_ν helicopter tip vortices cannot be modeled by the laminar formulations of the past. Instead one has to seek analytical representations of the phenomenon where at least the most fundamental effects of turbulence are included. Previous, theoretically more involved developments on the subject are those of Newman [8], Inversen [9], Tang [10], and Ramasamy and Leishman [7]. Here we present a new simple mathematically convenient formulation that accounts for the flattening effect in the tangential velocity profile.

Theoretical Development

Fifteen years ago, a family of laminar vortices was proposed by Vatistas et al. [11]. The most widely used member of the set is the $n = 2$. Recently, the basic model was transformed to account for vortex decay [12]. The formulation was further enlarged to include the effects of density variations [13]. The present paper represents the first attempt toward capturing the overall effects of turbulence via a mathematically simple model. A probable reason for the historical preference to Scully's [14] formulation in modeling tip vortices is also identified.

Subsequent to a great deal of theoretical deliberations, we have finally modified the original $n = 2$ laminar eddy into

$$V = \frac{V_\theta}{V_{\theta\max}} = \xi\left(\frac{\alpha+1}{\alpha+\xi^4}\right)^m \tag{1}$$

where V_θ and $V_{\theta\max}$ are the local and core tangential velocities, respectively, $\xi = r/r_{\max}$, r is the radial coordinate, $r_{\max}$ is the core size, α and m are scaling constants.

The requirement that V must be maximum (in fact 1) at $\xi = 1$ yields

$$m = \frac{\alpha+1}{4}$$

It is important to note that when $\alpha = 1$ the original $n = 2$ vortex formulation, applicable to laminar vortices, is recovered.

The value of α is found via the *least-squares method* by minimizing E,

$$E = \sum_{j=1}^{N}\left[V(\xi_j) - \xi_j\left(\frac{\alpha+1}{\alpha+\xi_j^4}\right)^{\frac{\alpha+1}{4}}\right]^2 \tag{2}$$

or

$$\sum_{j=1}^{N}\frac{\partial}{\partial\alpha}\left[V(\xi_j) - \xi_j\left(\frac{\alpha+1}{\alpha+\xi_j^4}\right)^{\frac{\alpha+1}{4}}\right] = 0$$

The ability of Eq. (1) to capture turbulent vortices is demonstrated in Fig. 1 using three sets of experimental data of relatively high Re_ν

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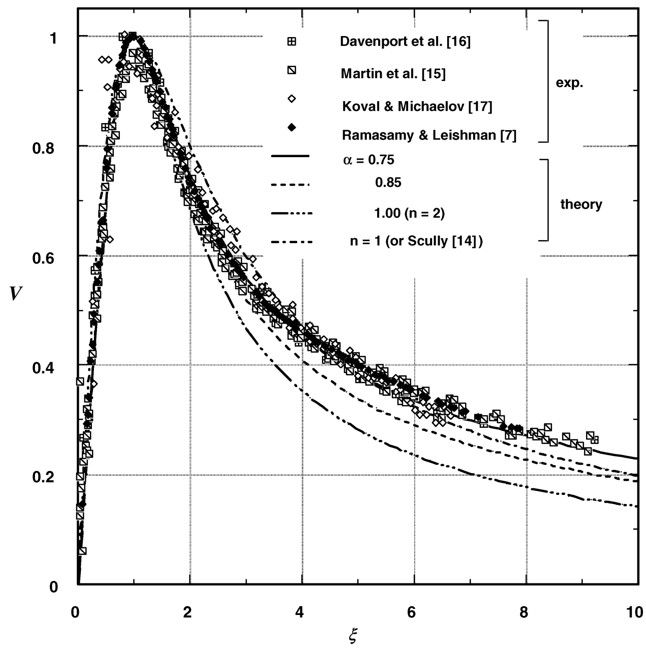

Fig. 1 Tangential velocity profiles for vortices with varying degrees of turbulence intensity.

tip vortices [7,15,16], and one set pertaining to a swirl chamber [17]. Inside the core, all curves approximate reasonably the laminar core. Right after the core the curve given by Eq. (1) begins to veer away from the laminar vortex profile ($\alpha = 1$). As the radius increases it progressively produces the anticipated flatter distribution. Parameter α represents the level of turbulence; the smaller the value of α (corresponding to larger Re_v), the higher the intensity of turbulence. Eventually, when a substantially large set of high-fidelity experimental observations, with different degree of turbulence are produced, then one can relate with confidence α to the vortex Reynolds number. The theoretical approach of Ramasamy and Leishman [7] yields results that are very comparable to the present. The turbulent vortex representations by Inversen [9] and Tang [10] appear to only approximate the phenomenon at higher radii. In order not to overcrowd Fig. 1 we do not include the curves of the previously mentioned models. However, one can find these in Ramasamy and Leishman [7].

At first, the present theoretical approach appears to be completely empirical. This, however, is not entirely true. It has been previously shown the system of equations that describe intense laminar vortices to be underdetermined [18]. Consequently, one has to specify one of the fluid parameters and the rest are obtained from the simplified Navier–Stokes group. If the classical models, which obey the same set, are considered exact solutions, so are the formulations of Scully [14] and Vatistas et al. [11]. Here we assume that the tangential velocity form is given by Eq. (1), along with the additional hypothesis; turbulence is roughly approximated by a uniformly enhanced value for the viscosity $\lambda_{\mathrm{eff}} = 1 + \nu_{\mathrm{lam}}/\nu_{\mathrm{tur}}$, where ν_{lam} and ν_{tur} are the laminar and turbulent kinematic viscosity values. Under this approximation the steady set of equations given in Vatistas et al. [12] is still applicable. It is a straightforward matter to then obtain the corresponding expressions for the other fluid properties, such as the other velocity components, the pressure, vorticity, etc.

In the past Rankine's, Burger's, or Scully's vortices were used to approximate tip vortices with the latter one being the preferred choice for tip vortices. But why should a formulation that clearly underestimates the tangential velocity in its natural form be the favored model? The answer may lie in the following. The derivative ratio (β) for the $n = 2$ and 1 models, given by

$$\beta = \left(\frac{dV}{d\xi}\right)_{n=2} \Bigg/ \left(\frac{dV}{d\xi}\right)_{n=1} = \frac{V_\theta}{V_{\theta\max}} = \frac{1}{\sqrt{2}}\left(1 + \frac{2\xi^2}{1+\xi^4}\right)^{3/2}$$

or

$$\left(\frac{dV}{d\xi}\right)_{n=2} \geq \left(\frac{dV}{d\xi}\right)_{n=1} \quad \text{for } 0 \leq \xi < \infty, \quad \text{with } \lim_{\substack{\xi\to\infty \\ \xi\to 0}} \beta = \frac{1}{\sqrt{2}}$$

is greater than one for all radii in $(0, \infty)$. This implies that for ξ values more than the core the normalized $n = 1$ (or Scully's) appears flatter than the $n = 2$ (or Burger's), which is the particular characteristic of turbulent vortices. Referring to Fig. 1, it is evident that the normalized Scully's vortex approximates better turbulent tip vortices in the region $\xi > \sim 3$ than the laminar models. The previous made argument might explain the preferred choice of the past.

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